Statistical mechanics of qp-bosons in D dimensions

M. Daoud and M. Kibler

Institut de Physique Nucliaire de Lyon,
IN2P3-CNRS et Universiti Claude Bernard,
43 Bd. du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

      This paper is concerned with statistical properties of a gas of qp-bosons without interaction. Some thermodynamical functions for such a system in D dimensions are derived. Bose-Einstein condensation is discussed in terms of the parameters q and p. Finally, the second-order correlation function of a gas of photons is calculated.

1. In the last twenty years, different types of deformed bosons have been introduced [1-8]. There use for obtaining realisations of quantum algebras is now widely widespread (e.g., see [4-6,8]). We shall deal here with qp-bosons [6-8] associated to the annihilation (a), creation (a^+) and number operator (N) characterized through their action

$$a|n\rangle = \sqrt{[[n]]_{qp}}|n-1\rangle, \quad a^+|n\rangle = \sqrt{[[n+1]]_{qp}}|n+1\rangle, \quad N|n\rangle = n|n\rangle \quad (1)$$

on the ordinary Fock space $\mathcal{F} := \{|n\rangle \ : \ n \in \mathbf{N}\}$. (The operators a, a^+ and N span a qp-deformed Weyl-Heisenberg algebra.) The qp-deformed number in (1) are defined by

$$[[x]]_{qp} := \frac{q^x - p^x}{q - p}. \quad (2)$$

The particular cases $p = 1$ and $p = q^{-1}$ correspond to q-deformations mainly used in the mathematical and physical literature, respectively. In the general case, Hermitean conjugation requirements demand that $(q,p) \in \mathbf{R} \times \mathbf{R}$, or $(q,p) \in \mathbf{C} \times \mathbf{C}$ with $p = \bar{q}$, or $(q,p) \in S^1 \times S^1$ with $p = \bar{q} = q^{-1}$ [9].

Numerous works have been devoted to the study of the statistical thermodynamical properties of a gas of deformed bosons (mainly q-bosons) without interaction [10-21]. (When writing this Letter, we became aware of a paper [21] dealing with qp-bosons.) In particular, the phenomenon of Bose-Einstein condensation of deformed bosons has been studied by several authors [10,13,14,16,20,21]. Furthermore, the deformation of the correlation function of order two for a gas of photons has been recently investigated by Man'ko et $al.$ [15]. Most of the preceding works concern q- and qp-bosons in $D = 2$ or 3 dimensions and are restricted to $q \in \mathbf{R}$ [10-20] and $(q,p) \in \mathbf{R} \times \mathbf{R}$ [21].

We derive in this Letter the (thermodynamical) properties of a gas of qp-bosons in D dimensions for parameters q and p varying in five domains.

The Bose-Einstein condensation of qp-bosons is discussed in a new approach. Finally, we obtain the correlation function $g^{(2)}$ for a gas of photons described by qp-bosons.

2. We start with the Hamiltonian (in a second-quantized form)

$$H := \sum_k H_k, \quad H_k := (E_k - \mu)N_k, \tag{3}$$

which describes a gas of qp-bosons without interaction. In eq. (3), μ is the chemical potential while E_k and N_k are the kinetic energy of a qp-boson and the number operator for the qp-bosons, in the k mode, respectively. The qp-analogue of the Bose factor for the k mode is then

$$(f_k)_{qp} := \frac{1}{Z}\mathrm{tr}\big(\mathrm{e}^{-\beta H} a_k^+ a_k\big), \tag{4}$$

where $Z := \mathrm{tr}\big(\mathrm{e}^{-\beta H}\big)$ is the partition function and $\beta = (k_B T)^{-1}$ the reciprocal temperature. As a trivial result, we have

$$Z = \prod_k \frac{1}{1 - \mathrm{e}^{-\eta}}, \quad \eta := \beta(E_k - \mu). \tag{5}$$

The partition function Z is thus independent of the deformation parameters q and p. Then, the Bose factor $(f_k)_{qp}$ reads

$$(f_k)_{qp} = (1 - \mathrm{e}^{-\eta})\mathrm{tr}\big(\mathrm{e}^{-\eta N_k}[[N_k]]_{qp}\big), \tag{6a}$$

which can be shown to converge in each of the following five cases : (i) $(q,p) \in \mathbf{R}^+ \times \mathbf{R}^+$ with $0 < q < \mathrm{e}^{-\beta\mu}$ and $0 < p < \mathrm{e}^{-\beta\mu}$; (ii) $(q,p) \in \mathbf{R}^+ \times \mathbf{R}^+$ with $p = q^{-1}$ and $\mathrm{e}^{\beta\mu} < q < \mathrm{e}^{-\beta\mu}$; (iii) $(q,p) \in \mathbf{R}^+ \times \mathbf{R}^+$ with $p = 1$ and $0 < q < \mathrm{e}^{-\beta\mu}$; (iv) $(q,p) \in \mathbf{C} \times \mathbf{C}$ with $p = \bar{q}$ and $0 < |q| < \mathrm{e}^{-\beta\mu}$; (v) $(q,p) \in S^1 \times S^1$ with $p = \bar{q} = q^{-1}$.

At this stage, it is important to justify the choices made for eqs. (3) and (4). The qp-bosons are not new particles. They are still bosons. Thus, (3) and (4) correspond to the basic formulas for the statistical mechanics of bosons. Here, we do not deform the basic formulas. The trace and the exponential in eq. (4) are nondeformed functions. We only use (through the formalism of second quantization) creation and annihilation operators that satisfy qp-deformed commutation relations. The form chosen for H_k in eq. (3) is nothing but the usual one where the number operator takes its eigenvalues in $\mathbf{N}$. The deformation cannot enter the theory at this level by replacing N_k by $a_k^+ a_k$ in (3) since we would obtain an "occupation number" $a_k^+ a_k = [[N_k]]_{qp}$ that would not have its eigenvalues in $\mathbf{N}$. On the contrary, we must keep $a_k^+ a_k$ rather than N_k in (4). Indeed, should we use N_k in place of $a_k^+ a_k$ in (4), we would get a nondeformed value of the Bose factor. As a conclusion, eqs. (3) and (4) constitute a reasonable starting point for a (minimal) deformation of the Bose distribution.

Following ref. [9], we can use the parametrisation

$$q = \mathrm{e}^{\varphi \cos \tau} \mathrm{e}^{+\varphi \sin \tau}, \quad p = \mathrm{e}^{\varphi \cos \tau} \mathrm{e}^{-\varphi \sin \tau} \tag{7}$$

for $(q, p) \in \mathbf{R} \times \mathbf{R}$ and the parametrisation

$$q = \mathrm{e}^{\varphi \cos \tau} \mathrm{e}^{+\mathrm{i}\varphi \sin \tau}, \quad p = \mathrm{e}^{\varphi \cos \tau} \mathrm{e}^{-\mathrm{i}\varphi \sin \tau} \tag{8}$$

for $(q, p) \in \mathbf{C} \times \mathbf{C}$ or $(q, p) \in S^1 \times S^1$, where $\varphi \in \mathbf{R}^*$ and $-\pi/2 \leq \tau \leq \pi/2$, so that the limiting situation $q = p = 1$ corresponds to $\varphi \to 0$.

The obtained expression for $(f_k)_{qp}$ is

$$(f_k)_{qp} = \frac{q-1}{q-p} \frac{1}{\mathrm{e}^\eta - q} + \frac{p-1}{p-q} \frac{1}{\mathrm{e}^\eta - p} \tag{9a}$$

or alternatively

$$(f_k)_{qp} = \frac{\mathrm{e}^\eta - 1}{(\mathrm{e}^\eta - q)(\mathrm{e}^\eta - p)}, \tag{10}$$

which clearly exhibits the $q \leftrightarrow p$ symmetry. In the (five) limiting situations where $\varphi \to 0$, we recover the ordinary Bose distribution $f_k = (e^\eta - 1)^{-1}$. Note that $(f_k)_{qp}$ can be rewritten as

$$(f_k)_{qp} = \frac{q-1}{q-p}\frac{f_k}{1+(1-q)f_k} + \frac{p-1}{p-q}\frac{f_k}{1+(1-p)f_k} \qquad (9b)$$

in term of f_k.

For the purpose of practical applications (in Section 3), it is useful to know the development of the distribution $(f_k)_{qp}$ in integer series. In this respect, we have

$$(f_k)_{qp} = \sum_{j=0}^{\infty} e^{-\eta(j+1)} \left(\frac{q-1}{q-p}q^j + \frac{p-1}{p-q}p^j \right), \qquad (6b)$$

which reduces to the expansion $\sum_{j=1}^{\infty} e^{-\eta j}$ of the ordinary Bose factor when $\varphi \to 0$.

3. In the thermodynamical limit, the energy spectrum of the system of qp-bosons may be considered as a continuum. Thus, $(f_k)_{qp}$ is replaced by the qp-dependent factor $f(\epsilon)$ which is $(f_k)_{qp}$ with $E_k = \varepsilon$. Therefore, in physical applications, we have to consider integrals of the type

$$J_s := \int_0^\infty \varepsilon^s f(\varepsilon) d\varepsilon. \qquad (11)$$

By using the development (6b), eq. (11) yields

$$J_s = \Gamma(s+1)(k_B T)^{s+1} \sigma(s+1)_{qp}, \quad s > -1, \qquad (12)$$

where

$$\sigma(s+1)_{qp} = \sum_{j=0}^{\infty} \frac{e^{\beta\mu(j+1)}}{(j+1)^{s+1}} \left([[j+1]]_{qp} - [[j]]_{qp} \right). \qquad (13)$$

[In eq. (12), Γ is the Euler integral of the second type.] In the case where $\mu = 0$, we have $\sigma(s+1)_{qp} \to \zeta(s+1)$ for $\varphi \to 0$, where $\zeta(s+1)$ is the Riemann Zeta series (that converges for $s > 0$).

We are now in a position to derive some thermodynamical quantities, in D dimensions, for a free gas of $N(D)$ qp-bosons contained in a volume $V(D)$. The density $\rho(D) = N(D)/V(D)$ is given by [14]

$$\rho(D) = N_0(D)\Gamma(\frac{D}{2})(k_B T)^{\frac{D}{2}}\sigma(\frac{D}{2})_{qp}, \tag{14}$$

where

$$N_0(D) := \frac{1}{2(2\pi)^{\frac{D}{2}}}\frac{D}{\Gamma(\frac{D}{2}+1)}g\frac{m^{\frac{D}{2}}}{\hbar^D}. \tag{15}$$

In eq. (15), g is the degree of (spin) degeneracy and m the mass of a qp-boson. The total energy can then be calculated to be

$$E(D) = N_0(D)V(D)\Gamma\left(\frac{D}{2}+1\right)(k_B T)^{\frac{D}{2}+1}\sigma\left(\frac{D}{2}+1\right)_{qp}. \tag{16}$$

The specific heat at constant volume easily follows from $C_V = \left(\frac{\partial E}{\partial T}\right)_V$. We obtain

$$C_V(D) = \frac{D}{2}N(D)k_B\left[\left(\frac{D}{2}+1\right)\frac{\sigma\left(\frac{D}{2}+1\right)_{qp}}{\sigma\left(\frac{D}{2}\right)_{qp}} - \frac{D}{2}\frac{\sigma\left(\frac{D}{2}\right)_{qp}}{\sigma\left(\frac{D}{2}-1\right)_{qp}}\right]. \tag{17}$$

Furthermore, the entropy is

$$S(D) = N(D)k_B\left[\left(\frac{D}{2}+1\right)\frac{\sigma\left(\frac{D}{2}+1\right)_{qp}}{\sigma\left(\frac{D}{2}\right)_{qp}} - \frac{\mu}{k_B T}\right]. \tag{18}$$

Note that the state equation for the gas of qp-bosons is

$$p(D) = \frac{2}{D}\frac{E(D)}{V(D)}, \tag{19}$$

so that the pressure $p(D)$ assumes the same form as in the nondeformed case.

Finally, other thermodynamical quantities may be determined in a simple manner by using the thermodynamic potential

$$\Omega(D) = -\frac{2}{D} N_0(D) V(D) J_{\frac{D}{2}} = -\frac{2}{D} E(D), \tag{20}$$

where $J_{\frac{D}{2}}$ is given by (12) and (13).

4. We now examine the condensation of a system of qp-bosons in D dimensions. The corresponding density for such a system is given by (14) which can be rewritten as

$$\rho(D) = N_0(D) J_{\frac{D}{2}-1} \tag{21}$$

in function of the integral $J_{\frac{D}{2}-1}$. As in the case of ordinary bosons, we define the Bose temperature $T_B(D)$ for the qp-bosons by taking $\mu = 0$. By introducing $\mu = 0$ in $J_{\frac{D}{2}-1}$, we find that the temperature below which we obtain a Bose-Einstein condensation is given by

$$T_B(D) = \frac{1}{k_B} \left[\frac{\rho(D)}{N_0(D)} \frac{1}{\Gamma(\frac{D}{2})\sigma_0(\frac{D}{2})_{qp}} \right]^{\frac{2}{D}}, \tag{22}$$

where

$$\sigma_0(\frac{D}{2})_{qp} = \sum_{j=0}^{\infty} \frac{1}{(j+1)^{\frac{D}{2}}} \left([[j+1]]_{qp} - [[j]]_{qp} \right). \tag{23}$$

The study of the convergence of the series (23) determines if Bose-Einstein condensation takes place or not.

In the limiting cases where $\varphi \to 0$, eqs. (22) and (23) give back the well-known results according to which there is (respectively, there is not)

condensation in $D = 3$ (respectively, $D = 2$) dimensions. As an illustration, we recover that the Bose temperature is

$$T_B(3) = \frac{(2\pi)\hbar^2}{k_B m} \left[\frac{\rho(3)}{2.612g} \right]^{\frac{2}{3}} \tag{24}$$

in the case $D = 3$.

Returning to the general case, we can prove the following results valid for D arbitrary.

(i) Case $(q, p) \in \mathbf{R} \times \mathbf{R}$: the series (23) converges for $0 < q < 1$ and $0 < p < 1$. Thus, there is Bose-Einstein condensation of qp-bosons for $0 < q < 1$ and $0 < p < 1$. In addition, there is no Bose-Einstein condensation (i.e., $T_B(D) = 0$) for q-bosons corresponding to $p = q^{-1}$. In contradistinction, Bose-Einstein condensation takes place for q-bosons corresponding to $p = 1$ when $0 < q < 1$.

(ii) Case $(q, p) \in \mathbf{C} \times \mathbf{C}$: Bose-Einstein condensation occurs either for the q-bosons corresponding to $p = \bar{q}$ when $0 < |q| < 1$ or for all q-bosons corresponding to $p = \bar{q} = q^{-1}$.

These results have to be compared to the ones corresponding to the limiting situation where $\varphi \to 0$ for which Bose-Einstein condensation exists only when $D \geq 3$.

5. As a final facet of this Letter, we would like to discuss the consequence of a qp-deformation of the correlation function $g^{(2)}$ of order two associated to the radiation field. In the nondeformed case, we know that $g^{(2)}$ takes two values, viz., $g^{(2)} = 1$ for a coherent monomode radiation and $g^{(2)} = 2$ for a chaotic monomode radiation. An interesting question arises : Is it possible to interpolate between the latter two values by replacing the ordinary bosons of the radiation field by qp-bosons ?

Our basic hypothesis is to describe the radiation field by an assembly of qp-bosons with the Hamiltonian

$$h = \sum_k h_k, \quad h_k = \hbar\omega_k N_k. \tag{25}$$

The corresponding Bose statistical distribution for the k mode is then given by (10) where η is replaced by $\xi = \beta\hbar\omega_k$. In the case of qp-bosons, we adopt the definition

$$g^{(2)} = \frac{\langle a^+ a^+ a a \rangle}{\langle a^+ a \rangle^2}, \tag{26}$$

where a and a^+ stand for the annihilation and creation operators for the k mode. The abbreviation $\langle X \rangle$ in (26) denotes the mean statistical value $Z^{-1}\text{tr}\left(e^{-\beta h} X\right)$ for an operator X defined on the Fock space $\mathcal{F}$. Equation (26) can be developed as

$$g^{(2)} = p^{-1}\frac{\langle (a^+ a)^2 \rangle}{\langle a^+ a \rangle^2} - (qp)^{-1}\frac{\langle q^N a^+ a \rangle}{\langle a^+ a \rangle^2}. \tag{27}$$

Of course, $\langle a^+ a \rangle = (f_k)_{qp}$ as given by (10) with $\eta \equiv \xi$. In addition, the other average values in (27) can be calculated to be

$$\langle (a^+ a)^2 \rangle = \frac{(e^\xi - 1)(e^\xi + qp)}{(e^\xi - q^2)(e^\xi - qp)(e^\xi - p^2)} \tag{28}$$

and

$$\langle q^N a^+ a \rangle = q\frac{e^\xi - 1}{(e^\xi - q^2)(e^\xi - qp)}. \tag{29}$$

Finally, we obtain

$$g^{(2)} = (q+p)\frac{1}{e^\xi - 1}\frac{(e^\xi - q)^2(e^\xi - p)^2}{(e^\xi - q^2)(e^\xi - qp)(e^\xi - p^2)}, \tag{30}$$

with convergence conditions that parallel the ones for $(f_k)_{qp}$, cf. cases (i)-(v) in Section 1 where $-\beta\mu$ must be replaced by $\xi/2$. [For instance, when

$(q, p) \in \mathbf{R}^+ \times \mathbf{R}^+$, we must have $0 < q < e^{\frac{\xi}{2}}$ and $0 < p < e^{\frac{\xi}{2}}$.] The qp-deformed factor $g^{(2)}$ depends on the parameters q and p in a symmetrical manner ($q \leftrightarrow p$ symmetry). It also presents a dependence on the energy $\hbar\omega_k$ of the k mode and on the temperature T. In the limiting cases $\varphi \to 0$, we get $g^{(2)} = 2$ that turns out to be the value of the correlation function for the choatic monomode radiation of the black body. Let us now examine the cases of low temperatures and high energies. In these cases, e^{ξ} is the dominating term in each of the differences occuring in eq. (30). Therefore, we have

$$g^{(2)} \sim q + p \tag{31}$$

at low temperature or high energy. It is thus possible to reach the value $g^{(2)} = 1$ without employing coherent states. Equation (31) shows that we can interpolate in a continuous way from $g^{(2)} = 1$ (coherent phase) to $g^{(2)} = 2$ (chaotic phase). It should be observed that we can even obtain either $g^{(2)} > 2$ or $g^{(2)} < 1$ from eq. (30). The situation where $g^{(2)} < 1$ may be interesting for describing antibunching effects of the light field (arising from the "nonclassical" nature of the radiation field, cf. ref. [21]).

6. In this work, we concentrated on the influence of a qp-deformation of the Weyl-Heisenberg algebra on the statistical properties of a gas of bosons in D dimensions. We discussed the phenomenon of Bose-Einstein condensation in terms of the deformation parameters q and p. Bose-Einstein condensation occurs in most of the domains where the q and p parameters may vary. In the case $(q, p) \in \mathbf{R}^+ \times \mathbf{R}^+$ with $p = q^{-1}$ and D arbitrary, Bose-Einstein condensation does not take place. For the latter case, we may ask the question : Are there any topological interactions (leading to some quasi-fermionic behavior through some exclusion rule) that prevent the condensation of the corresponding q-bosons ?

An important result of the present work concerns the correlation function $g^{(2)}$ for a gas of photons. The qp-deformation considered here makes it possible to obtain a continuum of values for $g^{(2)}$. In particular, such values may be greater than 2 or lesser than 1.

To close this Letter, it should be observed that alternative choices, non-linear in N_k, are possible for the Hamiltonians H (eq. (3)) and h (eq. (25)), choices that seem to reflect more deeply the quantum group structure of the qp-deformed Weyl-Heisenberg algebra. In this direction, we may mention, in the case $p = q^{-1}$, the works of refs. [11,12,15,19,20]. We have justified above our choice for eq. (3) (and eq. (4)). The case of eq. (25) is more delicate. It would be interesting to investigate the situation where $h_k = (1/2)\hbar\omega_k([[N_k]]_{qp} + [[N_k + 1]]_{qp})$ since the latter expression corresponds to a qp-deformed oscillator. The corresponding partition function can then be expressed in series of Bessel functions with some convergence problem. Our choice for eq. (25) does not lead to convergence problem and represents the first attempt to derive a deformation of the second-order correlation function in a nonperturbative way. We hope to return on these matters in the future.

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